

# Inverse problems for self-adjoint Dirac systems: explicit solutions and stability of the procedure

A.L. Sakhnovich

*To the memory of Leiba Rodman, a wonderful  
mathematician and an admirable person.*

## Abstract

A procedure to recover explicitly self-adjoint matrix Dirac systems on semi-axis (with both discrete and continuous components of spectrum) from rational Weyl functions is considered. Its stability is proved. GBDT version of Bäcklund-Darboux transformation and various important results on Riccati equations are used for this purpose.

MSC(2010): 15A24, 34A55, 34B20, 34D20, 93B20.

Keywords: *Inverse problem, stability, Dirac system, Weyl function, minimal realization, explicit solution, Riccati equation.*

## 1 Introduction

Self-adjoint Dirac system has the form

$$\frac{d}{dx}y(x, z) = i(zj + jV(x))y(x, z), \quad x \geq 0, \quad (1.1)$$

where

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \quad m_1 + m_2 =: m, \quad (1.2)$$

$I_{m_k}$  is the  $m_k \times m_k$  identity matrix and  $v(x)$  is an  $m_1 \times m_2$  matrix function. We assume that the *potential*  $v$  is locally summable (i.e., summable on all the finite intervals  $[0, l]$ ).

Inverse spectral problem for general-type self-adjoint Dirac system, and closely related problem to recover Dirac system from its Weyl-Titchmarsh (Weyl) function, had been actively studied since 1950s [28, 32] and many interesting results were published last years (see, e.g., [2, 3, 6, 7, 10, 23, 35, 43] and various references therein). Further, speaking about inverse spectral problems, we mean (in particular) the recovery of systems from their Weyl functions. Inverse spectral problems (in the mentioned above sense) are also solved [12, 14, 37, 39, 43] for general-type skew-self-adjoint and discrete Dirac systems.

Procedures to solve these inverse problems are nonlinear and usually unstable. However, stability plays an essential role in theory and applications, and several special cases where such stability could be proved are important. In particular, we could mention the paper [34] on the evolution Schrödinger equation and the paper [23], where stability was proved for a class of scalar ( $m_1 = m_2 = 1$ ) Dirac systems on interval with discrete and  $d$ -separated spectral data. Here, we consider the case of explicit solutions of inverse problems (i.e., the case of rational Weyl functions), and one can apply procedures, which are different from the procedures for the general-type case.

Thus, we prove stability of solving inverse problems for matrix Dirac systems on semi-axis with both discrete and continuous components of spectrum. Riccati equations play an essential role in the explicit solving of inverse problems, and so the classical works on Riccati equations by Leiba Rodman and coauthors are actively used in this paper.

Explicit solutions of inverse spectral problems can be obtained either by applying procedures for general-type systems to some specific spectral data (e.g., to rational Weyl or scattering functions) or by using several specific for explicit solutions procedures. The first (general-type) approach was used, for instance, in [1, 16, 42] and [2, Sect. 6]. The second approach includes Crum-Krein method [8, 29], commutation methods [9, 17, 18, 27] and some versions of Bäcklund-Darboux transformation. Here we use our GBDT version of the Bäcklund-Darboux transformation (see [38, 40, 43] and references therein),

see also [4, 5, 33, 48] and references therein on various versions of Bäcklund–Darboux transformations.

In the next section, Preliminaries, we present some basic notions from system theory and formulate several results on Weyl functions. We also present GBDT procedure to solve inverse problem for systems (1.1) (more precisely, to recover self-adjoint Dirac systems from Weyl functions). Section 3 is dedicated to the proof of stability of this procedure.

As usual,  $\mathbb{R}$  stands for the real axis,  $\mathbb{C}$  stands for the complex plane,  $\mathbb{C}_+$  is the open upper half-plane  $\{z : \Im(z) > 0\}$ ,  $\mathbb{C}_-$  is the open lower half-plane  $\{z : \Im(z) < 0\}$ , and the notation  $\text{diag}\{d_1, \dots\}$  stands for the diagonal (or block diagonal) matrix with the entries  $d_1, \dots$  on the main diagonal. By  $\|A\|$  and by  $\sigma(A)$ , we denote the  $l^2$ -induced norm and the spectrum, respectively, of some matrix  $A$ . We say that the matrix  $X$  is positive (positive definite) and write  $X > 0$  if  $X$  is Hermitian (i.e.,  $X = X^*$ ) and all the eigenvalues of  $X$  are positive.

## 2 Preliminaries

### 2.1 Rational functions

Recall that a rational matrix function is called *strictly proper* if it tends to zero at infinity. It is well-known [25, 30] that such an  $m_2 \times m_1$  matrix function  $\varphi$  can be represented in the form

$$\varphi(z) = \mathcal{C}(zI_n - \mathcal{A})^{-1}\mathcal{B}, \quad (2.1)$$

where  $\mathcal{A}$  is a square matrix of some order  $n$ , and the matrices  $\mathcal{B}$  and  $\mathcal{C}$  are of sizes  $n \times m_1$  and  $m_2 \times n$ , respectively. The representation (2.1) is called a *realization* of  $\varphi$ , and the realization (2.1) is said to be *minimal* if  $n$  is minimal among all possible realizations of  $\varphi$ . This minimal  $n$  is called the *McMillan degree* of  $\varphi$ . The realization (2.1) of  $\varphi$  is minimal if and only if

$$\text{span} \bigcup_{k=0}^{n-1} \text{Im } \mathcal{A}^k \mathcal{B} = \mathbb{C}^n, \quad \text{span} \bigcup_{k=0}^{n-1} \text{Im } (\mathcal{A}^*)^k \mathcal{C}^* = \mathbb{C}^n, \quad n = \text{ord}(\mathcal{A}), \quad (2.2)$$

where  $\text{Im}$  stands for image and  $\text{ord}(\mathcal{A})$  stands for the order of  $\mathcal{A}$ . If for a pair of matrices  $\{\mathcal{A}, \mathcal{B}\}$  the first equality in (2.2) holds, then the pair  $\{\mathcal{A}, \mathcal{B}\}$  is

called *controllable*. If the second equality in (2.2) is fulfilled, then the pair  $\{\mathcal{C}, \mathcal{A}\}$  is said to be *observable*.

Now, let matrix functions  $\varphi$  be contractive, that is, let  $\varphi^* \varphi \leq I_{m_1}$  (or, equivalently,  $\varphi \varphi^* \leq I_{m_2}$ ) hold. From [30, Theorems 21.1.3, 21.2.1] (see also [13, p. 191]), the next proposition easily follows.

**Proposition 2.1** *Assume that  $\varphi$  is a strictly proper rational matrix function, which is contractive on  $\mathbb{R}$  and has no poles in  $\mathbb{C}_+$ , and let realization (2.1) be its minimal realization.*

*Then, there is a positive solution  $X > 0$  of the Riccati equation*

$$X\mathcal{B}\mathcal{B}^*X + i(\mathcal{A}^*X - X\mathcal{A}) + \mathcal{C}^*\mathcal{C} = 0. \quad (2.3)$$

Clearly, under conditions of Proposition 2.1,  $\varphi(z)$  is contractive on  $\mathbb{C}_+ \cup \mathbb{R}$ .

In the case of the skew-self-adjoint Dirac system, we obtain Riccati equation with minus before  $\mathcal{B}\mathcal{B}^*$ :

$$X\mathcal{C}^*\mathcal{C}X + i(\mathcal{A}X - X\mathcal{A}^*) - \mathcal{B}\mathcal{B}^* = 0. \quad (2.4)$$

This case should be dealt with in a different way and we shall do it separately (in the next paper).

## 2.2 System (1.1): Weyl function and inverse problem

Recall that  $Y(x, z)$  is the normalized by  $Y(0, z) = I_m$  fundamental solution of Dirac system (1.1), where  $j$  and  $V$  have the forms (1.2).

**Definition 2.2** *An  $m_2 \times m_1$  matrix function  $\varphi(z)$  ( $z \in \mathbb{C}_+$ ) such that*

$$\int_0^\infty [I_{m_1} \quad \varphi(z)^*] Y(x, z)^* Y(x, z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty \quad (2.5)$$

*is called a Weyl function of the Dirac system (1.1) on  $[0, \infty)$ .*

**Remark 2.3** *According to [13, Sect. 2] and [43, Sect. 2.2], the Weyl function  $\varphi(z)$  of the Dirac system (1.1) always exists and is unique. Moreover,  $\varphi(z)$  is holomorphic and contractive in  $\mathbb{C}_+$ .*

If  $\varphi$  is rational, it can be prolonged (from  $\mathbb{C}_+$ ) on  $\mathbb{R}$  and  $\mathbb{C}_-$  in a natural way. Each potential  $v$  corresponding to a strictly proper rational Weyl function is generated by a fixed value  $n \in \mathbb{N}$  and by a quadruple of matrices, namely, by two  $n \times n$  matrices  $\alpha$  and  $S_0 > 0$  and by  $n \times m_k$  matrices  $\vartheta_k$  ( $k = 1, 2$ ) such that the matrix identity

$$\alpha S_0 - S_0 \alpha^* = i(\vartheta_1 \vartheta_1^* - \vartheta_2 \vartheta_2^*) \quad (2.6)$$

holds. Such potentials  $v$  have the form

$$v(x) = -2i\vartheta_1^* e^{ix\alpha^*} S(x)^{-1} e^{ix\alpha} \vartheta_2, \quad (2.7)$$

$$S(x) = S_0 + \int_0^x \Lambda(t) \Lambda(t)^* dt, \quad \Lambda(x) = [e^{-ix\alpha} \vartheta_1 \quad e^{ix\alpha} \vartheta_2]. \quad (2.8)$$

**Definition 2.4** [13, 19] *The potentials  $v$  generated by the quadruples  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$  (where  $S_0 > 0$  and (2.6) holds) via equalities (2.7) and (2.8), are called pseudo-exponential.*

**Theorem 2.5** [13] *Let Dirac system with a pseudo-exponential potential  $v$  be given on  $[0, \infty)$  and let  $v$  be generated by the quadruple  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ . Then the Weyl function  $\varphi$  of this system has the form*

$$\varphi(z) = -i\vartheta_2^* S_0^{-1} (zI_n - \theta)^{-1} \vartheta_1, \quad \theta = \alpha - i\vartheta_1 \vartheta_2^* S_0^{-1}. \quad (2.9)$$

The following theorem (i.e., [13, Theorem 3.4]) presents a procedure of explicit solution of the inverse problem (see also [22, Theorem 5.2] for the  $m_1 = m_2$  case), which is basic for this paper.

**Theorem 2.6** *Let  $\varphi(z)$  be a strictly proper rational matrix function, which is contractive on  $\mathbb{R}$  and has no poles in  $\mathbb{C}_+$ . Assume that (2.1) is its minimal realization and that  $X > 0$  is a solution of (2.3).*

*Then  $\varphi(z)$  is the Weyl function of the Dirac system (1.1), the potential  $v$  of which has the form (2.7), (2.8), where the quadruple  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$  is given (in terms of  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  and  $X$ ) by the relations*

$$\alpha = \mathcal{A} + i\mathcal{B}\mathcal{B}^*X, \quad S_0 = X^{-1}, \quad \vartheta_1 = \mathcal{B}, \quad \vartheta_2 = -iX^{-1}\mathcal{C}^*. \quad (2.10)$$

In particular, the identity (2.6) easily follows from (2.3) and (2.10). The uniqueness of our explicit solution of the inverse problem is immediate from a much more general uniqueness result.

**Proposition 2.7** *[41] The solution of the inverse problem to recover system (1.1) from its Weyl function is unique in the class of Dirac systems with the locally square summable potentials.*

**Remark 2.8** *We note that there are many quadruples generating the same pseudo-exponential potential. The quadruples, which are recovered using (2.10), have an important additional property: controllability of the pair  $\{\alpha, \vartheta_1\}$ . This property is immediate from the controllability of the pair  $\{\mathcal{A}, \mathcal{B}\}$ .*

Furthermore, the matrices  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  in the minimal realizations (2.1) of  $\varphi$  are unique up to *basis* (similarity) transformations:

$$\widehat{\mathcal{A}} = \mathcal{T}^{-1} \mathcal{A} \mathcal{T}, \quad \widehat{\mathcal{C}} = \mathcal{C} \mathcal{T}, \quad \widehat{\mathcal{B}} = \mathcal{T}^{-1} \mathcal{B}, \quad (2.11)$$

where  $\mathcal{T}$  are invertible  $m \times m$  matrices. Choosing the realization of  $\varphi$  with  $\widehat{\mathcal{A}}$ ,  $\widehat{\mathcal{B}}$  and  $\widehat{\mathcal{C}}$  instead of  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$ , and adding the sign "˜" in the notations of the corresponding matrices  $\alpha$ ,  $\vartheta_i$  and  $X$ , we derive

$$\widehat{\alpha} = \mathcal{T}^{-1} \alpha \mathcal{T}, \quad \widehat{\vartheta}_i = \mathcal{T}^{-1} \vartheta_i \quad (i = 1, 2), \quad \widehat{X} = \mathcal{T}^* X \mathcal{T}. \quad (2.12)$$

Setting  $\mathcal{T} = X^{-1/2} U^*$ , where  $U$  is unitary, we have  $\widehat{X} = I_m$ . Hence, (2.6) takes the form  $\widehat{\alpha} - \widehat{\alpha}^* = i(\widehat{\vartheta}_1 \widehat{\vartheta}_1^* - \widehat{\vartheta}_2 \widehat{\vartheta}_2^*)$ . Moreover, for the case  $m_1 = m_2 = p$ , it was shown in [21] that  $U$  may be chosen in such a way that we have the block representations:

$$\widehat{\beta} := \widehat{\alpha} - i\widehat{\vartheta}_1(\widehat{\vartheta}_1 + \widehat{\vartheta}_2)^* = \begin{bmatrix} \widetilde{\beta} & 0 \\ 0 & \zeta \end{bmatrix}, \quad \widehat{\vartheta}_1 = \begin{bmatrix} \widetilde{\vartheta}_1 \\ \omega \end{bmatrix}, \quad \widehat{\vartheta}_2 = \begin{bmatrix} \widetilde{\vartheta}_2 \\ -\omega \end{bmatrix}, \quad (2.13)$$

where

$$\zeta = \zeta^* = \text{diag}\{t_1 I_{n_1}, t_2 I_{n_2}, \dots, t_k I_{n_k}\}, \quad \sigma(\widetilde{\beta}) \in \mathbb{C}_-, \quad (2.14)$$

and  $\omega$  is some  $\widetilde{n} \times p$  matrix,  $\widetilde{n} := \sum_{i=1}^k n_k$ .

Now, introduce Dirac operator  $H$  associated with the differential expression

$$(\mathcal{H}f)(x) = \left( -\mathrm{i}j \frac{d}{dx} - V(x) \right) f(x), \quad (2.15)$$

the domain of which consists of all absolutely continuous functions  $f$  from  $L_m^2(0, \infty)$ , such that  $\mathcal{H}f \in L_m^2(0, \infty)$  and the initial condition

$$[I_p \quad -I_p] f(0) = 0$$

holds. Using (2.13), it is shown in [21] (see also [22, Sect. 2]) that the real eigenvalues of  $H$  are concentrated at the points  $t_k$  and have multiplicities  $n_k$ , whereas the continuous spectrum of  $H$  is described by  $\tilde{\beta}$ ,  $\tilde{\vartheta}_1$  and  $\tilde{\vartheta}_2$ . Namely, the spectral density  $\varrho$  of  $H$  has the form

$$\varrho(t) = g(t)^* g(t), \quad g(t) := I_p - \mathrm{i}(\tilde{\vartheta}_1 + \tilde{\vartheta}_2)^*(tI_{n-\tilde{n}} - \tilde{\beta})^{-1} \tilde{\vartheta}_1. \quad (2.16)$$

In view of the mentioned above connections between the quadruple  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$  and the corresponding Weyl and spectral functions, we can consider this quadruple as some spectral data.

### 3 Stability of explicit solutions

1. The following lemma is a stronger statement than Proposition 2.1. (Theorem 7.4.2 from [30] is used for its proof in addition to the Theorems 21.1.3 and 21.2.1 from [30], which yield Proposition 2.1.)

**Lemma 3.1** [15] *Assume that a strictly proper rational  $m_2 \times m_1$  matrix function  $\varphi(z)$  is contractive on  $\mathbb{R}$ , and that (2.1) is its minimal realization.*

*Then there is a unique Hermitian solution  $X$  of the Riccati equation (2.3) such that the relation*

$$\sigma(\mathcal{A} + \mathrm{i}\mathcal{B}\mathcal{B}^* X) \subset (\mathbb{C}_- \cup \mathbb{R}) \quad (3.1)$$

*holds. This solution  $X$  is always invertible. It is also positive if and only if  $\varphi(z)$  is contractive in  $\mathbb{C}_+$ .*

Further, in our procedure to recover the potential  $v$  from the Weyl function  $\varphi$ , we shall look for this particular solution  $X$  of (2.3). More precisely, we start with the strictly proper rational  $m_2 \times m_1$  matrix function  $\varphi(z)$ , which is contractive on  $\mathbb{R}$  and has no poles in  $\mathbb{C}_+$ . Hence,  $\varphi(z)$  is contractive in  $\mathbb{C}_+$ , and so, according to Lemma 3.1, we have  $X > 0$ . By  $\mathcal{G}_n$  we denote the class of triples  $\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}\}$  which determine minimal realizations  $\tilde{\varphi}(z) = \tilde{\mathcal{C}}(zI_n - \tilde{\mathcal{A}})^{-1}\tilde{\mathcal{B}}$  of  $m_2 \times m_1$  matrix functions  $\tilde{\varphi}(z)$  contractive on  $\mathbb{R} \cup \mathbb{C}_+$ . First, we consider the stability of recovery of  $X$  from  $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\} \in \mathcal{G}_n$ .

**Definition 3.2** *The recovery of  $X > 0$ , satisfying (3.1), from the minimal realization (2.1) (where  $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\} \in \mathcal{G}_n$ ) of  $\varphi(z)$  is called stable if for any  $\varepsilon > 0$  there is  $\delta > 0$  such that for each  $\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}\}$  satisfying conditions*

$$\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}\} \in \mathcal{G}_n, \quad \|\mathcal{A} - \tilde{\mathcal{A}}\| + \|\mathcal{B} - \tilde{\mathcal{B}}\| + \|\mathcal{C} - \tilde{\mathcal{C}}\| < \delta \quad (3.2)$$

*there is a solution  $\tilde{X} = \tilde{X}^*$  of the equation  $\tilde{X}\tilde{\mathcal{B}}\tilde{\mathcal{B}}^*\tilde{X} + i(\tilde{\mathcal{A}}^*\tilde{X} - \tilde{X}\tilde{\mathcal{A}}) + \tilde{\mathcal{C}}^*\tilde{\mathcal{C}} = 0$  in the neighborhood  $\|\tilde{X} - X\| < \varepsilon$  of  $X$ .*

The stability of solutions  $X$  of an important class of Riccati equations was shown in [36, Theorem 4.4] for a somewhat wider class of perturbations than described in our definition and we shall use this theorem in order to prove our first stability statement.

**Theorem 3.3** *The recovery of  $X > 0$ , satisfying (3.1), from the minimal realization (2.1) of  $\varphi(z)$  (with  $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\} \in \mathcal{G}_n$ ) is stable.*

**P r o o f.** Assuming that a minimal realization (2.1) of  $\varphi(z)$  is given (that is, matrices  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are given), we consider equation (2.3). Putting

$$A_0 = -i(\mathcal{A} + cI_n) \quad (c \in \mathbb{R}), \quad C_0 = -\mathcal{C}^*\mathcal{C}, \quad D_0 = \mathcal{B}\mathcal{B}^*, \quad (3.3)$$

we see that the equation (2.3) coincides with the Riccati equation  $XD_0X + XA_0 + A_0^*X - C_0 = 0$  considered in [36, Subsection 4.2].

Now, we deal with the conditions (i)-(iv) (on the coefficients  $A_0, C_0, D_0$ ) from [36, Subsection 4.2]. (Only perturbations satisfying these conditions are allowed in [36, Subsection 4.2] and we will show that the conditions (i)-(iv) are satisfied in the case  $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\} \in \mathcal{G}_n$ .) Equalities (3.3) and the

fact that the pair  $\{\mathcal{A}, \mathcal{B}\}$  is controllable imply that conditions (i) and (ii) in [36, Subsection 4.2] are fulfilled. In a similar way we derive that conditions (i) and (ii) are fulfilled for the Riccati equations  $\tilde{X}\tilde{D}_0\tilde{X} + \tilde{X}\tilde{A}_0 + \tilde{A}_0^*\tilde{X} - \tilde{C}_0 = 0$  corresponding to all the triples  $\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}\} \in \mathcal{G}_n$ . For sufficiently large values of  $|c|$ , the requirement (iii) that the matrix

$$H = \begin{bmatrix} -C_0 & A_0^* \\ A_0 & D_0 \end{bmatrix} \quad (3.4)$$

satisfies the condition  $\det H \neq 0$  and that signature of  $H$  equals zero is also fulfilled. Clearly,  $c$  may be chosen so that (iii) is valid if we substitute the triple  $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$  with any triple  $\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}\} \in \mathcal{G}_n$  in some small neighborhood of  $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ . Finally, according to Lemma 3.1, there are hermitian solutions of equations  $\tilde{X}\tilde{D}_0\tilde{X} + \tilde{X}\tilde{A}_0 + \tilde{A}_0^*\tilde{X} - \tilde{C}_0 = 0$ , that is, condition (iv) holds. Since conditions (i)-(iv) from [36, Subsection 4.2] are fulfilled, the stability with respect to perturbations in the class  $\mathcal{G}_n$  will follow from the stability in the sense of [36, Theorem 4.4].

Using again Lemma 3.1, we choose the solution  $X > 0$  of (2.3) satisfying (3.1). It is immediate that one of the equivalent statements from [36, Theorem 4.4] is valid for our  $X$ . That is, according to (3.1) and (3.3), the equality  $\Im(\lambda) = 0$  holds for each  $\lambda$  from the set

$$\begin{aligned} \sigma(i(A_0 + D_0X)) \cap \sigma(-i(A_0^* + XD_0)) \\ = \sigma(\mathcal{A} + i\mathcal{B}\mathcal{B}^*X + cI_n) \cap \sigma((\mathcal{A} + i\mathcal{B}\mathcal{B}^*X + cI_n)^*), \end{aligned} \quad (3.5)$$

and so the statement (d) from [36, Theorem 4.4] holds. Therefore, by virtue of [36, Theorem 4.4],  $X$  is a stable and isolated solution of (2.3). ■

**2.** Now, we will show that small perturbations of the quadruple  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$  result in small perturbations of the corresponding potential  $v$ . We note that we consider only perturbations which do not change  $m_1, m_2$  and  $n$ .

**Definition 3.4** *The quadruple  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$  is called admissible if  $S_0 > 0$  and (2.6) holds, and it is called spectral if it is admissible, the pair  $\{\alpha, \vartheta_1\}$  is controllable and*

$$\sigma(\alpha) \subset (\mathbb{R} \cup \mathbb{C}_-). \quad (3.6)$$

**Remark 3.5** Theorem 2.5 and Remark 2.3 show that the Weyl function corresponding to any pseudo-exponential potential is rational and contractive. Then Theorem 2.6, Proposition 2.7 and Lemma 3.1 imply that this potential (uniquely recovered from the Weyl function) is generated, in particular, by a spectral quadruple. In other words, each pseudo-exponential potential is generated by some spectral quadruple.

**Theorem 3.6** Let a spectral quadruple  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$  be given. Then, for any  $\varepsilon > 0$  there is  $\delta > 0$  such that each pseudo-exponential potential  $\tilde{v}$  generated by an admissible quadruple  $\{\tilde{\alpha}, \tilde{S}_0, \tilde{\vartheta}_1, \tilde{\vartheta}_2\}$  satisfying condition

$$\|\alpha - \tilde{\alpha}\| + \|S_0 - \tilde{S}_0\| + \|\vartheta_1 - \tilde{\vartheta}_1\| + \|\vartheta_2 - \tilde{\vartheta}_2\| < \delta$$

belongs to the  $\varepsilon$ -neighborhood of  $v$  generated by  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ , that is,

$$\sup_{x \in [0, \infty)} \|v(x) - \tilde{v}(x)\| < \varepsilon. \quad (3.7)$$

In order to prove the theorem above we generalize (for the case when  $m_1$  does not necessarily equal  $m_2$  and  $S_0$  does not necessarily equal  $I_n$ ) some results from [22] on asymptotics of

$$Q(x) := S_0 + 2 \int_0^x e^{2it\alpha} \vartheta_2 \vartheta_2^* e^{-2it\alpha^*} dt. \quad (3.8)$$

**Lemma 3.7** The following relations are valid for a spectral quadruple  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$  :

$$\lim_{x \rightarrow \infty} Q(x)^{-1} = 0, \quad \lim_{x \rightarrow \infty} \|Q(x)^{-1} e^{2ix\alpha} \vartheta_2\| = 0. \quad (3.9)$$

**P r o o f.** The proof uses some steps from the proof of [22, Proposition 3.3]. Since  $Q(x)$  is increasing and is bounded below by  $S_0 > 0$ , there is a limit  $\kappa_Q := \lim_{x \rightarrow \infty} Q(x)^{-1}$ . Next, we prove that  $\kappa_Q = 0$ . From the definition (3.8) and identity (2.6) we derive

$$\begin{aligned} \alpha Q(x) - Q(x)\alpha^* &= \alpha S_0 - S_0 \alpha^* - i(e^{2ix\alpha} \vartheta_2 \vartheta_2^* e^{-2ix\alpha^*} - \vartheta_2 \vartheta_2^*) \\ &= i\vartheta_1 \vartheta_1^* - i e^{2ix\alpha} \vartheta_2 \vartheta_2^* e^{-2ix\alpha^*}. \end{aligned} \quad (3.10)$$

Multiplying (from both sides) the left-hand side and right-hand side of (3.10) by  $Q(x)^{-1}$ , we obtain

$$\begin{aligned} Q(x)^{-1}\alpha - \alpha^*Q(x)^{-1} - iQ(x)^{-1}\vartheta_1\vartheta_1^*Q(x)^{-1} \\ = -iQ(x)^{-1}e^{2ix\alpha}\vartheta_2\vartheta_2^*e^{-2ix\alpha^*}Q(x)^{-1}. \end{aligned} \quad (3.11)$$

Passing in (3.11) to the limit, we see that

$$\lim_{x \rightarrow \infty} Q(x)^{-1}e^{2ix\alpha}\vartheta_2\vartheta_2^*e^{-2ix\alpha^*}Q(x)^{-1} = i(\kappa_Q\alpha - \alpha^*\kappa_Q - i\kappa_Q\vartheta_1\vartheta_1^*\kappa_Q). \quad (3.12)$$

On the other hand, formula (3.8) yields

$$\frac{d}{dx}Q(x)^{-1} = -2Q(x)^{-1}e^{2ix\alpha}\vartheta_2\vartheta_2^*e^{-2ix\alpha^*}Q(x)^{-1},$$

and so we have

$$\int_0^\infty Q(t)^{-1}e^{2it\alpha}\vartheta_2\vartheta_2^*e^{-2it\alpha^*}Q(t)^{-1}dt = \frac{1}{2}(S_0^{-1} - \kappa_Q) < \infty. \quad (3.13)$$

Taking into account (3.13) and the fact that there exists a limit of the expression integrated in (3.13) (see (3.12)), we derive that this limit equals zero. That is, we rewrite (3.12) in the form

$$\kappa_Q\alpha - \alpha^*\kappa_Q - i\kappa_Q\vartheta_1\vartheta_1^*\kappa_Q = 0. \quad (3.14)$$

Moreover, since the left-hand side in (3.12) tends to zero, the second equality in (3.9) is already proved.

Recall that the first equality in (3.9) is equivalent to  $\kappa_Q = 0$ . Now, we prove  $\kappa_Q = 0$  by negation. For this, we rewrite (3.14) in the form  $\alpha^*\kappa_Q = \kappa_Q(\alpha - i\vartheta_1\vartheta_1^*\kappa_Q)$ , which implies that the range of  $\kappa_Q$  is an invariant subspace of  $\alpha^*$ . Thus, assuming  $\kappa_Q \neq 0$ , we obtain that there is an eigenvector  $\kappa_Qg$  of  $\alpha^*$ :  $\alpha^*\kappa_Qg = c\kappa_Qg$ ,  $\kappa_Qg \neq 0$ ,  $g \in \mathbb{C}^n$ .

Finally, consider the expression  $ig^*(\kappa_Q\alpha - \alpha^*\kappa_Q)g$ . In view of (3.6), for the eigenvalue  $c$  of  $\alpha^*$  we have  $\Im(c) \geq 0$ , and so

$$ig^*(\kappa_Q\alpha - \alpha^*\kappa_Q)g = i(\bar{c} - c)g^*\kappa_Qg \geq 0.$$

On the other hand, we have  $\vartheta_1^* \kappa_Q g \neq 0$  because the pair  $\{\alpha, \vartheta_1\}$  is controllable. Hence, the inequality  $ig^*(\kappa_Q \alpha - \alpha^* \kappa_Q)g = -g^* \kappa_Q \vartheta_1 \vartheta_1^* \kappa_Q g < 0$  follows from (3.14). We arrive at a contradiction, that is,  $\kappa_Q = 0$ .  $\blacksquare$

In the case of admissible quadruples  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ , the matrix identity

$$\alpha S(x) - S(x)\alpha^* = i\Lambda(x)j\Lambda(x)^* \quad (3.15)$$

(see [13, formula (3.6)]) coincides with (2.6) at  $x = 0$  and easily follows from (2.6) and (2.8) for  $x > 0$ . In other words,  $\alpha$ ,  $S(x)$  and  $\Lambda(x)$  form an  $S$ -node (and, moreover, the so called Darboux matrix function corresponding to  $v(x)$  coincides with the transfer matrix function [43–45] in Lev Sakhnovich sense). Using (2.8), (3.8) and (3.15), we derive

$$Q'(x) = (e^{ix\alpha} S(x) e^{-ix\alpha^*})', \quad Q(0) = S(0) \quad \left( Q' := \frac{d}{dx} Q \right),$$

and so the following equality is valid:

$$Q(x) = e^{ix\alpha} S(x) e^{-ix\alpha^*}. \quad (3.16)$$

*Proof of Theorem 3.6.* Now, we consider a pseudo-exponential potential  $v$  generated by the spectral quadruple  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$  and pseudo-exponential potentials  $\tilde{v}$  generated by admissible quadruples  $\{\tilde{\alpha}, \tilde{S}_0, \tilde{\vartheta}_1, \tilde{\vartheta}_2\}$  belonging to a neighborhood of  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ . The matrix function  $Q$  corresponding to  $\{\tilde{\alpha}, \tilde{S}_0, \tilde{\vartheta}_1, \tilde{\vartheta}_2\}$  is denoted by  $\tilde{Q}$ . In view of (2.7) and (3.16), we have:

$$v(x) = -2i\vartheta_1^* Q(x)^{-1} e^{2ix\alpha} \vartheta_2, \quad \tilde{v}(x) = -2i\tilde{\vartheta}_1^* \tilde{Q}(x)^{-1} e^{2ix\tilde{\alpha}} \tilde{\vartheta}_2. \quad (3.17)$$

It is immediate from the proof of Lemma 3.7 that (3.11) holds for admissible quadruples as well. That is, we may rewrite (3.11) for  $\{\tilde{\alpha}, \tilde{S}_0, \tilde{\vartheta}_1, \tilde{\vartheta}_2\}$ :

$$\begin{aligned} \tilde{Q}(x)^{-1} \tilde{\alpha} - \tilde{\alpha}^* \tilde{Q}(x)^{-1} - i\tilde{Q}(x)^{-1} \tilde{\vartheta}_1 \tilde{\vartheta}_1^* \tilde{Q}(x)^{-1} \\ = -i\tilde{Q}(x)^{-1} e^{2ix\tilde{\alpha}} \tilde{\vartheta}_2 \tilde{\vartheta}_2^* e^{-2ix\tilde{\alpha}^*} \tilde{Q}(x)^{-1}. \end{aligned} \quad (3.18)$$

Since  $Q$  and  $\tilde{Q}$  are monotonic and the first relation in (3.9) is valid, we may choose  $x_0 > 0$  and some neighborhood of  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$  so that  $Q(x)$  and  $\tilde{Q}(x)$  are large enough for  $x \geq x_0$ , and so the left-hand sides of (3.11) and

(3.18) are small enough. Hence, the right-hand sides of (3.11) and (3.18) are also small enough. Therefore, taking into account (3.17), we see that for any  $\varepsilon > 0$  there are  $x_0 > 0$  and  $\delta_1 > 0$  such that the next inequality holds in the  $\delta_1$ -neighborhood ( $\|\alpha - \tilde{\alpha}\| + \|S_0 - \tilde{S}_0\| + \|\vartheta_1 - \tilde{\vartheta}_1\| + \|\vartheta_2 - \tilde{\vartheta}_2\| < \delta_1$ ) of  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ :

$$\sup_{x \in [x_0, \infty)} \|v(x) - \tilde{v}(x)\| < \varepsilon. \quad (3.19)$$

It easily follows from the definition of  $Q$  and  $\tilde{Q}$  and from (3.17) that there is some  $\delta_2$ -neighborhood of  $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ , where we have

$$\sup_{x \in [0, x_0)} \|v(x) - \tilde{v}(x)\| < \varepsilon. \quad (3.20)$$

Clearly, inequalities (3.19) and (3.20) yield (3.7) (for  $\delta = \min(\delta_1, \delta_2)$ ). ■

**Remark 3.8** *It is immediate from the second relation in (3.9), the first relation in (3.17) and Remark 3.5 that all pseudo-exponential potentials tend to zero at infinity.*

**3.** Theorems 2.6, 3.3 and 3.6 as well as Lemma 3.1 yield the result below on the stability of the procedure of solving inverse problem.

**Theorem 3.9** *The procedure (given in Theorem 2.6) to uniquely recover the pseudo-exponential potential  $v$  of Dirac system (1.1) from a minimal realization of the Weyl function (i.e., of some strictly proper rational  $m_2 \times m_1$  matrix function, which is contractive in  $\mathbb{C}_+$ ) is stable once we agree to choose such a positive solution  $X$  of the Riccati equation (2.3) that (3.1) holds.*

**Acknowledgments.** This research was supported by the Austrian Science Fund (FWF) under Grant No. P24301.

## References

- [1] D. ALPAY AND I. GOHBERG, *Inverse spectral problem for differential operators with rational scattering matrix functions*, J. Differential Equations **118** (1995), 1–19.
- [2] D. ALPAY, I. GOHBERG, M.A. KAASHOEK, L. LERER, AND A. SAKHNOVICH, *Krein systems and canonical systems on a finite interval: accelerants with a jump discontinuity at the origin and continuous potentials*, Integral Equations Operator Theory **68**:1 (2010), 115–150.
- [3] R. BRUNNHUBER, J. ECKHARDT, A. KOSTENKO, AND G. TESCHL, *Singular Weyl–Titchmarsh–Kodaira theory for one-dimensional Dirac operators*, Monatsh. Math. **174** (2014), 515–547.
- [4] J.L. CIESLINSKI, *An algebraic method to construct the Darboux matrix*, J. Math. Phys. **36** (1995), 5670–5706.
- [5] J.L. CIESLINSKI, *Algebraic construction of the Darboux matrix revisited*, J. Phys. A **42** (2009), 404003.
- [6] S. CLARK AND F. GESZTESY, *Weyl–Titchmarsh M-function asymptotics, local uniqueness results, trace formulas, and Borg-type theorems for Dirac operators*, Trans. Amer. Math. Soc. **354** (2002), 3475–3534.
- [7] S. CLARK AND F. GESZTESY, *On Self-adjoint and  $J$ -self-adjoint Dirac-type operators: a case study*, Contemp. Math. **412** (2006), 103–140.
- [8] M.M. CRUM, *Associated Sturm–Liouville systems*, Quart. J. Math., Oxford II Ser. **6** (1955), 121–127.
- [9] P.A. DEIFT, *Applications of a commutation formula*, Duke Math. J. **45** (1978), 267–310.

- [10] J. ECKHARDT, F. GESZTESY, R. NICHOLS, A. SAKHNOVICH, AND G. TESCHL, *Inverse spectral problems for Schrödinger-type operators with distributional matrix-valued potentials*, Differential Integral Equations **28** (2015), 505–522.
- [11] B. FRITZSCHE, M.A. KAASHOEK, B. KIRSTEIN, AND A.L. SAKHNOVICH, *Skew-self-adjoint Dirac systems with rational rectangular Weyl functions: explicit solutions of direct and inverse problems and integrable wave equations*, arXiv:1501.00395.
- [12] B. FRITZSCHE, B. KIRSTEIN, I.YA. ROITBERG, AND A.L. SAKHNOVICH, *Skew-self-adjoint Dirac system with a rectangular matrix potential: Weyl theory, direct and inverse problems*, Integral Equations Operator Theory **74**:2 (2012), 163–187.
- [13] B. FRITZSCHE, B. KIRSTEIN, I.YA. ROITBERG, AND A.L. SAKHNOVICH, *Weyl theory and explicit solutions of direct and inverse problems for a Dirac system with rectangular matrix potential*, Oper. Matrices **7**:1 (2013), 183–196.
- [14] B. FRITZSCHE, B. KIRSTEIN, I.YA. ROITBERG, AND A.L. SAKHNOVICH, *Discrete Dirac system: rectangular Weyl functions, direct and inverse problems*, Oper. Matrices **8**:3 (2014), 799–819.
- [15] B. FRITZSCHE, B. KIRSTEIN AND A.L. SAKHNOVICH, *Completion problems and scattering problems for Dirac type differential equations with singularities*, J. Math. Anal. Appl. **317** (2006), 510–525.
- [16] B. FRITZSCHE, B. KIRSTEIN, AND A.L. SAKHNOVICH, *Semiseparable integral operators and explicit solution of an inverse problem for the skew-self-adjoint Dirac-type system*, Integral Equations Operator Theory **66** (2010), 231–251.
- [17] F. GESZTESY, *A complete spectral characterization of the double commutation method*, J. Funct. Anal. **117** (1993), 401–446.

- [18] F. GESZTESY AND G. TESCHL, *On the double commutation method*, Proc. Amer. Math. Soc. **124** (1996), 1831–1840.
- [19] I. GOHBERG, M.A. KAASHOEK, AND A.L. SAKHNOVICH, *Canonical systems with rational spectral densities: explicit formulas and applications*, Mathematische Nachr. **194** (1998), 93–125.
- [20] I. GOHBERG, M.A. KAASHOEK, AND A.L. SAKHNOVICH, *Pseudocanonical systems with rational Weyl functions: explicit formulas and applications*, J. Differential Equations **146**:2 (1998), 375–398.
- [21] I. GOHBERG, M.A. KAASHOEK, AND A.L. SAKHNOVICH, *Bound states for canonical systems on the half and full line: explicit formulas*, Integral Equations Operator Theory **40**:3 (2001), 268–277.
- [22] I. GOHBERG, M.A. KAASHOEK, AND A.L. SAKHNOVICH, *Scattering problems for a canonical system with a pseudo-exponential potential*, Asymptotic Analysis **29** (2002), 1–38.
- [23] R.O. HRYNIV, *Analyticity and uniform stability in the inverse spectral problem for Dirac operators*, J. Math. Phys. **52** (2011), 063513.
- [24] M.A. KAASHOEK AND A.L. SAKHNOVICH, *Discrete pseudo-canonical system and isotropic Heisenberg magnet*, J. Funct. Anal. **228** (2005), 207–233.
- [25] R.E. KALMAN, P. FALB, AND M. ARBIB, *Topics in mathematical system theory*, International Series in Pure and Applied Mathematics, McGraw-Hill Book Company, New York etc., 1969.
- [26] V.E. KATSNELSON, *Right and left joint system representation of a rational matrix function in general position*, in: D. ALPAY, ET AL (eds), *Operator theory, system theory and related topics*, pp. 337–400, Oper. Theory Adv. Appl. **123**, Birkhäuser, Basel, 2001.
- [27] A. KOSTENKO, A. SAKHNOVICH, AND G. TESCHL, *Commutation methods for Schrödinger operators with strongly singular potentials*, Math. Nachr. **285** (2012), 392–410.

- [28] M.G. KREIN, *Continuous analogues of propositions on polynomials orthogonal on the unit circle* (Russian), Dokl. Akad. Nauk SSSR **105** (1955), 637–640.
- [29] M.G. KREIN, *On a continuous analogue of a Christoffel formula from the theory of orthogonal polynomials* (Russian), Dokl. Akad. Nauk SSSR **113** (1957), 970–973.
- [30] P. LANCASTER AND L. RODMAN, *Algebraic Riccati equations*, Clarendon Press, Oxford, 1995.
- [31] H. LANGER, A.C.M. RAN, AND D. TEMME, *Nonnegative solutions of algebraic Riccati equations*, Lin. Alg. Appl. **261** (1997), 317–352.
- [32] B.M. LEVITAN AND I.S. SARGIJAN, *Introduction to the spectral theory. Selfadjoint differential operators*, Transl. Math. Monographs **34**, Amer. Math. Soc., Providence, RI, 1975.
- [33] V.B. MATVEEV AND M.A. SALLE, *Darboux transformations and solitons*, Springer, Berlin, 1991.
- [34] A. MERCADO, A. OSSES, AND L. ROSIER, *Carleman inequalities and inverse problems for the Schrödinger equation*, C. R. Math. Acad. Sci. Paris **346**:1-2 (2008), 53–58.
- [35] YA.V. MYKYTYUK AND D.V. PUYDA, *Inverse spectral problems for Dirac operators on a finite interval*, J. Math. Anal. Appl. **386** (2012), 177–194.
- [36] A.C.M. RAN AND L. RODMAN, *Stability of invariant maximal semidefinite subspaces, II: Applications: selfadjoint rational matrix functions, algebraic Riccati equations*, Linear Algebra Appl. **63** (1984), 133–173.
- [37] A.L. SAKHNOVICH, *Nonlinear Schrödinger equation on a semi-axis and an inverse problem associated with it*, Ukr. Math. J. **42**:3 (1990), 316–323.

- [38] A.L. SAKHNOVICH, *Exact solutions of nonlinear equations and the method of operator identities*, Lin. Alg. Appl. **182** (1993), 109–126.
- [39] A.L. SAKHNOVICH, *Skew-self-adjoint discrete and continuous Dirac-type systems: inverse problems and Borg–Marchenko theorems*, Inverse Problems **22** (2006), 2083–2101.
- [40] A.L. SAKHNOVICH, *On the GBDT version of the Bäcklund–Darboux transformation and its applications to linear and nonlinear equations and spectral theory*, Math. Model. Nat. Phenom. **5** (2010), 340–389.
- [41] A.L. SAKHNOVICH, *Inverse problem for Dirac systems with locally square-summable potentials and rectangular Weyl functions*, arXiv:1401.3605, Journal of Spectral Theory to appear.
- [42] A.L. SAKHNOVICH, A.A. KARELIN, J. SECK-TUOH-MORA, G. PEREZ-LECHUGA, AND M. GONZALEZ-HERNANDEZ, *On explicit inversion of a subclass of operators with  $D$ -difference kernels and Weyl theory of the corresponding canonical systems*, Positivity **14** (2010), 547–564.
- [43] A.L. SAKHNOVICH, L.A. SAKHNOVICH, AND I.YA. ROITBERG, *Inverse problems and nonlinear evolution equations. Solutions, Darboux matrices and Weyl–Titchmarsh functions*, De Gruyter Studies in Mathematics **47**, De Gruyter, Berlin, 2013.
- [44] L.A. SAKHNOVICH, *On the factorization of the transfer matrix function*, Sov. Math. Dokl. **17** (1976), 203–207.
- [45] L.A. SAKHNOVICH, *Factorisation problems and operator identities*, Russian Math. Surv. **41** (1986), 1–64.
- [46] B. SIMON, *Schrödinger operators in the twentieth century*, J. Math. Phys. **41** (2000), 3523–3555.
- [47] G. TESCHL, *Deforming the point spectra of one-dimensional Dirac operators*, Proc. Amer. Math. Soc. **126** (1998), 2873–2881.

[48] V.E. ZAKHAROV AND A.V. MIKHAILOV, *On the integrability of classical spinor models in two-dimensional space-time*, Commun. Math. Phys. **74** (1980), 21–40.

Alexander Sakhnovich,  
Vienna University of Technology, Austria,  
e-mail: [oleksandr.sakhnovych@tuwien.ac.at](mailto:oleksandr.sakhnovych@tuwien.ac.at)